

FRACTIONAL LOOP GROUP AND TWISTED K-THEORY

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ABSTRACT. We study the structure of abelian extensions of the group $L_q G$ of q -differentiable loops (in the Sobolev sense), generalizing from the case of central extension of the smooth loop group. This is motivated by the aim of understanding the problems with current algebras in higher dimensions. Highest weight modules are constructed for the Lie algebra. The construction is extended to the current algebra of supersymmetric Wess-Zumino-Witten model. An application to the twisted K-theory on G is discussed.

1. INTRODUCTION

The main motivation for the present paper comes from trying to understand the representation theory of groups of gauge transformations in higher dimensions than one. In the case of a circle, the relevant group is the loop group LG of smooth functions on the unit circle S^1 taking values in a compact Lie group G . In quantum field theory one considers representations of a central extension \widehat{LG} of LG ; in case when G is semisimple, this corresponds to an affine Lie algebra. The requirement that the energy is bounded from below leads to the study of highest weight representations of \widehat{LG} . This part of the representation theory is well understood, [6].

In higher dimensions much less is known. Quantum field theory gives us a candidate for an extension of the gauge group $Map(M, G)$, the group of smooth mappings from a compact manifold M to a compact group G . The extension is not central, but by an abelian ideal. The geometric reason for this is that the curvature form of complex line bundles (the Chern class of which is determined by a quantum anomaly) is not homogeneous; it is not invariant under left (or right) translations, [11].

There are two main obstructions when trying to extend the representation theory of affine Lie algebras to the case of $Map(M, \mathfrak{g})$. The first is that there is no natural polarization giving meaning to the highest weight condition; on S^1 the polarization is given by the decomposition of loops to positive and negative Fourier modes. The second obstruction has to do with renormalization problems in higher dimensions. On

the circle, with respect to the Fourier polarization, one can use methods of canonical quantization for producing representations of the loop group; the only renormalization needed is the normal ordering of quantities quadratic in the fermion field, [9], [17]. In higher dimensions further renormalization is needed, leading to an action of the gauge group, not in a single Hilbert space, but in a Hilbert bundle over the space of gauge connections, [12].

In this paper we make partial progress in trying to resolve the two obstructions above. We consider instead of LG the group L_qG of loops which are not smooth but only differentiable of order $0 < q \leq 1$ in the Sobolev sense, the *fractional loop group*. In the range $\frac{1}{2} \leq q$ the usual theory of highest weight representations is valid, the cocycle determining the central extension is well defined down to the critical order $q = \frac{1}{2}$. However, for $q < \frac{1}{2}$ we have to use again a “renormalized” cocycle defining an abelian extension of the group L_qG , similar as in the case of $Map(M, G)$. The renormalization means that the restriction of the 2-cocycle to the smooth subgroup $LG \subset L_qG$ is equal to the 2-cocycle c for the central extension (affine Kac-Moody algebra) plus a coboundary $\delta\eta$ of a 1-cochain η (the renormalization 1-cochain). The 1-cochain is defined only on LG and does not extend to L_qG when $q < 1/2$, only the sum $c + \delta\eta$ is well-defined on L_qG .

The important difference between L_qG and $Map(M, G)$ is that in the former case we still have a natural polarization of the Lie algebra into positive and negative Fourier modes and we can still talk about highest weight modules for the Lie algebra $L_q\mathfrak{g}$.

Because of the existence of the highest weight modules for $L_q\mathfrak{g}$ we can even define the supercharge operator Q for the supersymmetric Wess-Zumino-Witten model. In the case of the central extension the supercharge is defined as a product of a fermion field on the circle and the gauge current; this is well defined because the vacuum is annihilated both by the negative frequencies of the fermion field and the current, thus when acting on the vacuum only the (finite number of) zero Fourier modes remain. This property is still intact in the case of the abelian extension of the current algebra.

We can also introduce a family of supercharges, by a “minimal coupling” to a gauge connection on the loop group. In the case of central extension the connections on LG can be taken to be left invariant and they are written as a fixed connection plus a left invariant 1-form A on LG . The form A at the identity element is identified as a vector in the dual $L\mathfrak{g}^*$ which again is identified, through an invariant inner product, as a vector in $L\mathfrak{g}$ which in turn defines a \mathfrak{g} -valued 1-form on the circle. The left translations on LG induce the gauge action on the potentials A . Modulo the action of the group ΩG of based loops, the set of vector potentials on the circle is equal to

the group G of holonomies. In this way the family of supercharges parametrized by A defines an element in the twisted K-theory of G . Here the twist is equal to an integral 3-cohomology class on G fixed by the level k of the loop group representation, [13].

In the case of $L_q G$ and the abelian extension the connections are not invariant under the action of $L_q G$ and thus we have to consider the larger family of supercharges parametrized by the space \mathcal{A} of all connections of a circle bundle over $L_q G$. This is still an affine space, the extension of $L_q G$ acts on it. The family of supercharges transforms equivariantly under the extension and it follows that it can be viewed as an element in twisted K-theory of the moduli stack $\mathcal{A}/\widehat{L_q G}$. This replaces the G-equivariant twisted K-theory on \mathcal{A}/\widehat{LG} in case of the central extension \widehat{LG} , the latter being equivalent to twisted G-equivariant K-theory on the group G of gauge holonomies.

The paper is organized as follows. In Section 2 we introduce the fractional loop group and consider its role as the gauge group of a fractional Dirac-Yang-Mills system. It turns out that the natural setting is a spectral triple in the sense of non-commutative geometry. Interestingly enough, similar attempts have been made recently [4]. We move on to discuss the embedding $L_q G \subset GL_p$ in Section 3 and the construction of Lie algebra cocycles in Section 4. Finally the last two sections are devoted to extending the current algebra of the supersymmetric WZW model to the fractional case and discussing its application to the twisted K-theory of G .

2. FRACTIONAL LOOP GROUP

Let G denote a compact semisimple Lie group and \mathfrak{g} its Lie algebra. Fix a faithful representation $\rho : G \rightarrow GL(V)$ in a finite dimensional complex vector space V .

Definition 2.1. *The fractional loop group $L_q G$ for real index $\frac{1}{2} \leq q$ is defined to be the Sobolev space,*

$$L_q G := H^q(S^1, G) = \{g \in L^\infty(S^1, G) \mid \|g\|_{2,q}^2 = \sum_{k \in \mathbb{Z}} (1 + k^2)^q |\rho(g_k)|^2 < \infty\} ,$$

where $|\rho(g_k)|$ is the standard matrix norm of the k :th Fourier component of $g : S^1 \rightarrow G$ and L^∞ denotes the set of measurable essentially bounded functions. The group operation is given by pointwise multiplication $(g_1 g_2)(x) = g_1(x) g_2(x)$.

It is a Hilbert Lie group with the Lie algebra

$$L_q \mathfrak{g} = \{X \in L^\infty(S^1, \mathfrak{g}) \mid \|X\| = \|X\|_\infty + \|X\|_{2,q} < \infty\}$$

where $\|\cdot\|_\infty$ denotes the supremum norm [17]. We have a natural inclusion $LG \subset L_q G$ of the smooth loop group. In the next section, we will extend the definition of $L_q G$ to allow for values $0 < q < \frac{1}{2}$, with respect to a different norm.

In Yang-Mills theory on the cylinder $S^1 \times \mathbb{R}$, the loop group LG appears as the group of (time-independent) local gauge transformations. Quantization of massless chiral fermions in external Yang-Mills fields breaks the local gauge symmetry, leading to a central extension of LG . In order to make sense of this in the fractional setting, we must introduce the notion of fractional differentiation. The study of fractional calculus dates back to early 18th century and a comprehensive review can be found in [18]. The transition to fractional calculus is by no means unique. There are several competing definitions, but many are known to coincide on overlapping domains. For functions on the real line the Riemann-Liouville fractional derivative is defined by

$$D_a^q \psi(x) = \frac{d^n}{dx^n} \left\{ \frac{1}{\Gamma(n-q)} \int_a^x \frac{\psi(y)}{(x-y)^{q-n+1}} dy \right\}$$

for $n > q$ and $x > a$. On the circle however this definition proves inconvenient since periodic functions are not mapped into periodic ones. An operator that preserves periodicity is the Weyl fractional derivative,

$$\tilde{D}^q \psi(x) = \sum_{k \in \mathbb{Z}} (ik)^q \psi_k e^{ikx} = \sum_{k \in \mathbb{Z}} e^{\frac{iq\pi}{2} \text{sgn}(k)} |k|^q \psi_k e^{ikx}$$

for all $q \in \mathbb{R}$. In fact, by extending to the real line one can show that \tilde{D}^q and D_a^q coincide for $-1 < q$ and $a = -\infty$ on an appropriate domain. For our purposes, we need a self-adjoint operator on a dense domain in $\mathcal{H} = L^2(S^1, V)$. The fractional Dirac operator on the circle is therefore defined as

$$D^q \psi(x) = \sum_{k \in \mathbb{Z}} \text{sgn}(k) |k|^q \psi_k e^{ikx},$$

where the complex phase has been replaced by the sign function. This has the consequence that $D^q \circ D^r \neq D^{q+r}$, but instead we have $D^q \circ D^r = |D^{q+r}|$. The domain of D^q is the Sobolev space $H^q(S^1, V)$. It is by construction an unbounded, self-adjoint operator with discrete spectrum $\{\text{sgn}(k) |k|^q\}_{k \in \mathbb{Z}}$ and a complete set of eigenstates in \mathcal{H} . However, the Leibniz rule is no longer satisfied

$$D^q(\psi\phi) = \sum_{k, m \in \mathbb{Z}} \text{sgn}(k) |k|^q \psi_m \phi_{k-m} e^{ikx} \neq$$

$$\neq (D^q \psi)\phi + \psi(D^q \phi) = \sum_{k, m \in \mathbb{Z}} \left(\text{sgn}(m) |m|^q + \text{sgn}(k-m) |k-m|^q \right) \psi_m \phi_{k-m} e^{ikx},$$

unless $q = 1$.

We introduce interactions by imposing local gauge invariance. The covariant derivative $D_A^q = D^q + A$ should transform equivariantly under gauge transformations

$$g^{-1}D_A^q g = D^q + g^{-1}[D^q, g] + g^{-1}Ag = D_{Ag}^q .$$

This motivates the following definition of fractional Yang-Mills connections on the circle;

$$A = \alpha[D^q, \beta], \quad \alpha, \beta \in H^q(S^1, \mathfrak{g}) .$$

The fractional loop group $L_q G$ acts on A by

$$(g, A) \mapsto A^g = g^{-1}Ag + g^{-1}[D^q, g]$$

and the infinitesimal gauge action is simply

$$(X, A) \mapsto \mathcal{L}_X A = [A, X] + [D^q, X] .$$

There is a geometric interpretation of this data in the non-commutative geometry sense. What we have is precisely a spectral triple, namely a Dirac operator D^q , a Hilbert space \mathcal{H} and an associative $*$ -algebra $L_q \mathbb{C}$, [2].

Proposition 2.2. $[D^q, X]$ is a bounded operator for all $X \in L_q \mathbb{C}$ and $0 < q \leq 1$,

$$\|[D^q, X]\| = \sup_{\|\psi\|=1, \psi \in \mathcal{H}} \|[D^q, X]\psi\| < \infty .$$

Proof. By

$$[D^q, X]\psi = \sum_{k, m \in \mathbb{Z}} X_m \psi_{k-m} \left(\operatorname{sgn}(k)|k|^q - \operatorname{sgn}(k-m)|k-m|^q \right) e^{i(k+m)x}$$

it follows that

$$\begin{aligned} \|[D^q, X]\psi\|^2 &= \sum_{m, k \in \mathbb{Z}} |X_m \psi_{k-m}|^2 \left(\operatorname{sgn}(k)|k|^q - \operatorname{sgn}(k-m)|k-m|^q \right)^2 \\ &= \sum_{m, n \in \mathbb{Z}} |X_m \psi_n|^2 \left(\operatorname{sgn}(n+m)|n+m|^q - \operatorname{sgn}(n)|n|^q \right)^2 . \end{aligned}$$

Since the sequence $|\psi_n|^2$ belongs to l^1 , the sum converges if

$$\sum_{m \in \mathbb{Z}} |X_m|^2 \left(\operatorname{sgn}(n+m)|n+m|^q - \operatorname{sgn}(n)|n|^q \right)^2 < C$$

is uniformly bounded for all n by a constant C . To establish this, we rewrite

$$\sum_{m \in \mathbb{Z}} |X_m|^2 \left(\operatorname{sgn}(n+m)|n+m|^q - \operatorname{sgn}(n)|n|^q \right)^2 = \sum_{m \in \mathbb{Z}} |X_m|^2 m^{2q} \left(\operatorname{sgn} \left(1 + \frac{m}{n} \right) \left| 1 + \frac{m}{n} \right|^q - \left| \frac{n}{m} \right|^q \right)^2 .$$

This sum is bounded since

$$f(x) = \operatorname{sgn} \left(1 + \frac{1}{x} \right) |1 + x|^q - |x|^q$$

is a bounded function (for $0 < q \leq 1$) on the real line. \square

We have an immediate corollary:

Corollary 2.3. *For $\frac{1}{2} \leq q \leq 1$ the space $L_q\mathbb{C}$ is the algebra of essentially bounded measurable loops X such that $[D^q, X]$ is a bounded operator.*

The inverse statement follows from the observation that taking $\psi(x) = 1$, the constant loop in the proof above, the norm $\|[D^q, X]\psi\|$ is equal to $\|X\|_{2,q}$.

Similarly one can verify that $[|D^q|, X]$ is bounded for all $X \in L_q\mathbb{C}$. Recall that a spectral triple is p^+ -summable if $|D^q|^{-p}$ belongs to the Dixmier ideal \mathcal{L}^{1+} . This means that for some real number $p \geq 1$,

$$\lim_{N \rightarrow \infty} \frac{1}{\log(N)} \sum_{k=1}^N \lambda_k < \infty$$

where $\lambda_k \geq \lambda_{k+1} \geq \lambda_{k+2} \dots$ are eigenvalues of $|D^q|^{-p}$ listed in descending order. In our case

$$\lambda_k = \frac{1}{k^{qp}}$$

for $k = 1, 2, \dots$. This gives

$$\lim_{N \rightarrow \infty} \frac{1}{\log(N)} \sum_{k=1}^N \frac{1}{k^{qp}}$$

which is finite if and only if $qp \geq 1$. Moreover, since

$$\left\| \text{ad}_{|D|^q}^n(X)\psi \right\|^2 = \sum_{m,k \in \mathbb{Z}} |X_m \psi_{k-m}|^2 \left(|k|^q - |k-m|^q \right)^{2n}$$

diverges for $n > 1$, we conclude that the spectral triple is not tame.

It is also interesting to note that even though the algebra $L_q\mathbb{C}$ is commutative, the spectral dimension p of the circle is strictly larger than one when $q < 1$.

Fractional differentiability has been studied systematically in the more general context of θ -summable spectral triples in [5]. However, their definition of fractional differentiability, although similar, differs from ours which is geared to the special case of loop groups and L_p -summable spectral triples.

3. EMBEDDING OF L_qG IN GL_p

When dealing with representations of the loop group LG , one is lead to consider central extensions by the circle

$$1 \rightarrow S^1 \rightarrow \widehat{LG} \rightarrow LG \rightarrow 1 .$$

In Fourier basis, the generators S_n^a of the Lie algebra $\widehat{L\mathfrak{g}}$ satisfy

$$[S_n^a, S_m^b] = \lambda^{abc} S_{n+m}^c + kn \delta^{ab} \delta_{n,-m}$$

where k is the central element (represented as multiplication by a scalar in an irreducible representation). Here the upper index refers to a normalized basis (with respect to an invariant nondegenerate bilinear form) of the Lie algebra \mathfrak{g} of the group G . The λ^{abc} 's are the structure constants in this basis. We have adopted the Einstein summation convention meaning that an index appearing twice in a term is summed over all its possible values. When trying to extend this to the fractional setting, one runs immediately into a problem. For infinite linear combinations of the Fourier modes S_n^a the central term blows up. A precise condition for the divergence can be formulated by regarding $L_q G$ as a group of operators in a Hilbert space. Let $\rho : G \rightarrow GL(V)$ denote a representation of G as in the previous section. Elements in the fractional loop group act as multiplication operators in $\mathcal{H} = L^2(S^1, V)$ by pointwise multiplication,

$$(M_g \psi)(x) = \rho(g(x)) \psi(x)$$

for all $g \in L_q G$, $\psi \in \mathcal{H}$. In fact, $M : L_q G \rightarrow GL(\mathcal{H})$, $g \mapsto M_g$ defines a continuous embedding into the general linear group. This statement is somewhat crude however. Below we will show that $L_q G$ is actually contained in a subgroup GL_p of GL . Recall that the sign operator $\epsilon = \frac{D^q}{|D^q|}$ defines an orthogonal decomposition $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ into positive and negative Fourier modes. We introduce the Schatten class

$$L_{2p} = \{A \in \mathcal{B}(\mathcal{H}) \mid \|A\|_{2p} = [\text{Tr}(A^\dagger A)^p]^{\frac{1}{2p}} < \infty\}$$

which is a two-sided ideal in the algebra of bounded operators $\mathcal{B}(\mathcal{H})$. In particular, the first Schatten class L_1 is the space of trace class operators and L_2 is the space of Hilbert-Schmidt operators. We will use an equivalent norm which is better suited for computations

$$\|A\|_{2p} = \left[\sum_{k \in \mathbb{Z}} \|A \phi_k\|^{2p} \right]^{\frac{1}{2p}}$$

where $\{\phi_k\}_{k \in \mathbb{Z}}$ is an orthonormal basis in \mathcal{H} . The subgroup $GL_p \subset GL(\mathcal{H})$ is defined by

$$GL_p = \{A \in GL(\mathcal{H}) \mid [\epsilon, A] \in L_{2p}\}.$$

Writing elements in $GL(\mathcal{H})$ in block form with respect to the Hilbert space polarization,

$$A = \begin{pmatrix} A_{++} & A_{+-} \\ A_{-+} & A_{--} \end{pmatrix}$$

the condition

$$[\epsilon, A] = 2 \begin{pmatrix} 0 & A_{+-} \\ -A_{-+} & 0 \end{pmatrix} \in L_{2p}$$

simply means that the off-diagonal blocks are not “too large”. Given the topology defined by the norm

$$\|A_{++}\| + \|A_{+-}\|_{2p} + \|A_{-+}\|_{2p} + \|A_{--}\|$$

where

$$\|a\| = \sup_{\|\psi\|=1} \|a\psi\|,$$

GL_p is a Banach Lie group with the Lie algebra

$$\mathfrak{gl}_p = \{X \in \mathcal{B}(\mathcal{H}) \mid [\epsilon, X] \in L_{2p}\}.$$

Proposition 3.1. *If $p \geq \frac{1}{2q}$, then $L_q G$ is contained in GL_p .*

Proof. In order to avoid cumbersome notation, we write $g(x)$ instead of $\rho(g(x))$. Expanding in Fourier series $g(x) = \sum_{k \in \mathbb{Z}} g_k e^{ikx}$, we have

$$M_g e^{ikx} = \sum_{m \in \mathbb{Z}} g_m e^{i(m+k)x} = \sum_{m \in \mathbb{Z}} g_{m-k} e^{imx}$$

so that $(M_g)_{mk} = g_{m-k}$. Moreover

$$\epsilon e^{ikx} = \begin{cases} \frac{k}{|k|} e^{ikx} & k \neq 0 \\ e^{ikx} & k = 0 \end{cases}$$

We check that $\|[\epsilon, M_g]\|_{2p}$ is finite;

$$\begin{aligned} \|[\epsilon, M_g]\|_{2p}^{2p} &= \sum_{k \in \mathbb{Z}} \left\| [\epsilon, M_g] e^{ikx} \right\|^{2p} \\ &= \sum_{m, k \in \mathbb{Z}} \left| \frac{m}{|m|} - \frac{k}{|k|} \right|^{2p} |g_{k-m} e^{imx}|^{2p} \\ &= \sum_{n, k \in \mathbb{Z}} \left| \frac{(n+k)}{|n+k|} - \frac{k}{|k|} \right|^{2p} |g_n|^{2p} \\ &= 2^{2p} \sum_{n \in \mathbb{Z}} |n| |g_n|^{2p} \\ &= 2^{2p} \sum_{n \in \mathbb{Z}} |n|^{\frac{1}{2p}} |g_n|^{2p} \\ &\leq 2^{2p} \sum_{n \in \mathbb{Z}} (n^2)^{\frac{1}{2p}} |g_n|^2 + C_g \end{aligned}$$

where C_g is the finite part of the sum containing all terms where $|n|^{\frac{1}{2p}} |g_n| \geq 1$. Thus the sum converges if the Sobolev norm

$$\|g\|_{2, \frac{1}{2p}}^2 = \sum_{n \in \mathbb{Z}} (1 + n^2)^{\frac{1}{2p}} |g_n|^2$$

is finite, which is the case when

$$p \geq \frac{1}{2q} .$$

□

The same arguments apply without modification to the Lie algebra $L_q \mathfrak{g} \hookrightarrow \mathfrak{gl}_p$. We see here precisely how the degree of differentiability q is related to the Schatten index p . Furthermore, this allows us to extend the definition of $L_q G$ to all real values $0 < q < 1$:

Definition 3.2. *The fractional loop group $L_q G$ is defined to be the group continuous loops contained in $GL_{\frac{1}{2q}}$. We shall use the induced Banach-Lie structure in $L_q G$ coming from the embedding, so that in particular $L_q G \hookrightarrow GL_p$ is continuous for $p \geq \frac{1}{2q}$.*

The cocycle defining the central extension $\widehat{L\mathfrak{g}}$ can be written

$$c_0(X, Y) = \frac{1}{8} \text{Tr} \left(\epsilon [[\epsilon, X], [\epsilon, Y]] \right) = \text{Tr} \left(X_{+-} Y_{-+} - Y_{+-} X_{-+} \right)$$

for $X, Y \in L\mathfrak{g}$. This is finite so long as the off-diagonal blocks are Hilbert-Schmidt, i.e. $[\epsilon, X]$ and $[\epsilon, Y]$ belong to L_2 . However for $p > 1$, corresponding to differentiability q less than $\frac{1}{2}$, the operators $X_{+-} Y_{-+}$ and $Y_{+-} X_{-+}$ are no longer trace-class because $a, b \in L_{2p}$ implies that $ab \in L_p$. This is the reason behind the divergence. In order to make sense of the cocycle for higher p , regularization is required.

We introduce the Grassmannian Gr_p , which is a smooth Banach manifold parametrized by idempotent hermitian operators F such that $F - \epsilon \in L_{2p}$. Points on Gr_p may also be thought of as closed subspaces $W \subset \mathcal{H}$ such that the orthogonal projection $\text{pr}_+ : W \rightarrow \mathcal{H}_+$ is Fredholm and $\text{pr}_- : W \rightarrow \mathcal{H}_-$ is in L_{2p} . Let $\eta(X; F)$ denote a 1-cochain parametrized by points on Gr_p . For a suitable choice of η , adding the coboundary to the original cocycle

$$c_p(X, Y; F) = c_0(X, Y) + (\delta\eta)(X, Y; F)$$

we obtain a well-defined cocycle on \mathfrak{gl}_p for some fixed p . The Lie algebra cohomology coboundary operator δ is defined by Palais' formula in Section 4. Although each term on the right diverges separately, the sum will be finite. In the case $p = 2$, we could take

$$\eta(X; F) = -\frac{1}{16} \text{Tr} \left([X, \epsilon] [F, \epsilon] \right)$$

which gives

$$c_2(X, Y; F) = c_0(X, Y) + (\delta\eta)(X, Y; F) = \frac{1}{8} \text{Tr} \left([[\epsilon, X], [\epsilon, Y]] (\epsilon - F) \right) .$$

An important consequence of this so called “infinite charge renormalization” is that the central extension $\widehat{L\mathfrak{g}}$ is replaced by an abelian extension by the infinite-dimensional

ideal $\text{Map}(Gr_p, \mathbb{C})$,

$$0 \rightarrow \text{Map}(Gr_p, \mathbb{C}) \rightarrow \widehat{L_q \mathfrak{g}} \rightarrow L_q \mathfrak{g} \rightarrow 0 .$$

4. LIE ALGEBRA COCYCLES

In this section we will construct Lie algebra cocycles for \mathfrak{gl}_p for all $p \geq 1$, which by restriction yield cocycles on $L_q \mathfrak{g}$. In particular, we show that the cocycles respect the decomposition of $L_q \mathfrak{g} = L_q \mathfrak{g}_+ \oplus L_q \mathfrak{g}_-$ into positive and negative Fourier modes on the circle. The computation is done in the non-commutative BRST bicomplex, where the classical de Rham complex is replaced by a graded differential algebra (Ω, d) . Let $\epsilon = \frac{D^q}{|D^q|}$ denote the sign operator on the circle satisfying $\epsilon = \epsilon^*$ and $\epsilon^2 = 1$. We introduce a family of vector spaces

$$\begin{aligned} \Omega^0 &= \{X \in \mathcal{B}(\mathcal{H}) \mid X - \epsilon X \epsilon \in L_{2p}\} \\ \Omega^{2k-1} &= \{X \in \mathcal{B}(\mathcal{H}) \mid X + \epsilon X \epsilon \in L_{\frac{2p}{2k}}, X - \epsilon X \epsilon \in L_{\frac{2p}{2k-1}}\} \\ \Omega^{2k} &= \{X \in \mathcal{B}(\mathcal{H}) \mid X + \epsilon X \epsilon \in L_{\frac{2p}{2k}}, X - \epsilon X \epsilon \in L_{\frac{2p}{2k+1}}\} \end{aligned}$$

and set $\Omega = \bigoplus_{k=0}^{\infty} \Omega^k$. The exterior differentiation is defined by

$$dX = \begin{cases} [\epsilon, X], & \text{if } X \in \Omega^{2k} \\ \{\epsilon, X\}, & \text{if } X \in \Omega^{2k-1} \end{cases}$$

and satisfies $d^2 = 0$. Here $\{X, Y\} = XY + YX$ is the anticommutator. The space Ω^k consists of linear combinations of k -forms $X_0 dX_1 dX_2 \dots dX_k$ with $X_i \in \Omega^0$. If $X \in \Omega^k$ and $Y \in \Omega^l$, then

$$XY \in \Omega^{k+l}, \quad dX \in \Omega^{k+1}, \quad d(XY) = (dX)Y + (-1)^k X dY$$

which follows by the generalized Hölder inequality for Schatten ideals. This ensures that (Ω, d) is a \mathbb{N} -graded differential algebra. Integration of forms is substituted by a graded trace functional

$$\text{Str}(X) = \text{Tr}_C(\Gamma X), \quad \text{for } X \in \Omega^k \text{ and } k \geq p,$$

where Γ is a grading operator on \mathcal{H} and $\text{Tr}_C(X) = \frac{1}{2} \text{Tr}(X + \epsilon X \epsilon)$ is the conditional trace, [2]. In case of an even Fredholm module (k even), Γ anticommutes with ϵ and in case of an odd Fredholm module (k odd), $\Gamma = 1$.

Next we define the Lie algebra chain complex (C, δ) . Since $\Omega^0 = \mathfrak{gl}_p$ and $Gr_p \subset \Omega^1$, we interpret $B \in \Omega^1$ as generalized connection 1-forms and define the infinitesimal gauge action by

$$\Omega^0 \times \Omega^1 \rightarrow \Omega^1, \quad (X, B) \mapsto \mathcal{L}_X B = [B, X] + dX .$$

Indeed for any $F \in Gr_p$ we have $F = g^{-1}\epsilon g$ for some $g \in GL_p$, since GL_p acts transitively on the Grassmannian. Thus $F - \epsilon = g^{-1}dg$ corresponds to flat connections. The abelian group $Map(\Omega^1, \Omega)$ is naturally a Ω^0 -module under the action

$$\Omega^0 \times Map(\Omega^1, \Omega) \rightarrow Map(\Omega^1, \Omega), \quad (X, f) \mapsto \mathcal{L}_X f(B) = \frac{d}{dt} f\left(e^{-tX} B e^{tX} + t dX\right) \Big|_{t=0}.$$

Define the space of k -chains C^k as alternating multilinear maps

$$\omega : \underbrace{\Omega^0 \times \dots \times \Omega^0}_k \rightarrow Map(\Omega^1, \Omega)$$

and set $C = \bigoplus_{k=0}^{\infty} C^k$. The coboundary operator is given by Palais' formula

$$\begin{aligned} \delta\omega(X_1, \dots, X_k; B) &= \sum_{j=1}^k (-1)^{j+1} \mathcal{L}_{X_j} \omega(X_1, \dots, \hat{X}_j, \dots, X_k; B) \\ &\quad + \sum_{i < j} (-1)^{i+j} \omega([X_i, X_j], X_1, \dots, \hat{X}_i, \dots, \hat{X}_j, \dots, X_k; B) \end{aligned}$$

and satisfies $\delta^2 = 0$. Here \hat{X}_j means that the variable X_j is omitted.

This provides us with a double complex. For our purposes, we need cocycles of degree 2 in the Lie algebra cohomology. There is a straightforward way to compute such cocycles parametrized by *flat* connections $B = F - \epsilon$. Given a bicomplex with commuting differentials, there is an associated singly graded complex with differential $D = \delta + (-1)^k d$. The Lie algebra 2-cocycle is given by

$$\begin{aligned} \tilde{c}_p(X, Y; B) &= \frac{2^{2p}}{(2p+1)} \text{Str}(g^{-1} Dg)^{2p+3}_{[2p+1, 2]}(X, Y) \\ &= 2^{2p} \text{Str} \left(\sum_{k=0}^p (g^{-1} dg)^{2p+1-k} (g^{-1} \delta g) (g^{-1} dg)^k (g^{-1} \delta g) \right) (X, Y) \\ &= 2^{2p} \text{Str} \left(\sum_{k=0}^p (-1)^k (B^{2p+1-k} X B^k Y - B^{2p+1-k} Y B^k X) \right) \end{aligned}$$

for all $p \geq 0$. By $(\dots)_{[j, k]}$ we mean the component of degree (j, k) in (d, δ) cohomology. That this is a cocycle follows from

$$\delta \text{Str}(g^{-1} Dg)^{2p+3} = \text{Str} D(g^{-1} Dg)^{2p+3} = \text{Str}(g^{-1} Dg)^{2p+4} = 0$$

where we have used that $\text{Str}(g^{-1} Dg)^{\text{even}} = 0$. Although $\tilde{c}_p(X, Y; B)$ does not vanish when $X, Y \in L_q \mathfrak{g}_-$, we have

Theorem 4.1. *The cocycle \tilde{c}_p is cohomologous to a cocycle c_p which has the property that $c_p(X, Y; B) = 0$ if both X and Y are in $L_q \mathfrak{g}_-$ (or both in $L_q \mathfrak{g}_+$).*

Note that we can replace the abelian ideal of smooth functions of the variable B by the space of functions on $L_q G$ by using the embedding of $L_q G$ to the Grassmannian Gr_p by $g \mapsto B = g^{-1}[\epsilon, g]$.

The proof of the Theorem is by direct computation and is shifted to the Appendix.

Theorem 4.2. *The cocycle c_p in the Appendix, when restricted to the Lie algebra of smooth loops, is cohomologous to the cocycle defining the standard central extension of the loop algebra $L\mathfrak{g}$.*

Proof. Define $\eta_p(X; B) = 2^{2p+1} \text{Tr}(\epsilon B^{2p+1} dX)$. Then by direct computation, similar to the one in the Appendix and which will not be repeated here,

$$c_{p+1}(X, Y; B) = c_p(X, Y; B) - (\delta\eta_p)(X, Y; B) .$$

Thus c_p is cohomologous to c_{p-1} and by induction to the cocycle c_0 . But

$$c_0(X, Y; B) = \frac{1}{2} \text{Tr}(X dY)$$

which defines the central extension of the loop algebra. □

We note that the 1-cochain $\eta(X; F)$ in Section 3 is simply the sum $-\sum_{k=0}^{p-1} \eta_k(X; B)$, where $B = F - \epsilon$.

Remark Since the Lie algebra cocycle vanishes on $L_q \mathfrak{g}_-$ one can define generalized Verma modules as the quotient of the universal enveloping algebra $\mathcal{U}(\widehat{L_q \mathfrak{g}})$ by the left ideal generated by $L_q \mathfrak{g}_-$ and by the elements $h - \lambda(h)$ where the h 's are elements in a Cartan subalgebra \mathfrak{h} of \mathfrak{g} and $\lambda \in \mathfrak{h}^*$ is a weight. In the case of the central extension of the smooth loop algebra, for dominant integral weights one can construct an invariant hermitean form in the Verma module; there is a subquotient (including the highest weight vector) which carries an irreducible unitarizable representation of the Lie algebra. However, in the case of $L_q \mathfrak{g}$ for $q < 1/2$ this is not possible: due to the large abelian ideal in the extension of $L_q \mathfrak{g}$ we cannot construct any invariant hermitean semidefinite form on the Verma module. Whether this is possible at all is an open question, although we conjecture that the answer is negative.

5. GENERALIZED SUPERSYMMETRIC WZW MODEL

Let us recall the construction of a family of supercharges $Q(A)$ for the supersymmetric Wess-Zumino-Witten model in the setting of representations of the smooth loop algebra, [13], [8], [3].

Here we assume that G is a connected, simply connected simple compact Lie group of dimension N . Let \mathcal{H}_b denote the “bosonic” Hilbert space carrying an irreducible unitary highest weight representation of the loop algebra $\widehat{L\mathfrak{g}}$ of level k . The level is quantized to non-negative integer multiples of $\frac{2}{\theta^2}$, where θ is the length of the longest root of G . The generators of the loop algebra in Fourier basis T_n^a satisfy

$$[T_n^a, T_m^b] = \lambda^{abc} T_{n+m}^c + \frac{k}{4} n \delta^{ab} \delta_{n,-m}$$

where $n \in \mathbb{Z}$ and $a = 1, 2, \dots, N$. We fix an orthonormal basis T^a in \mathfrak{g} , with respect to the Killing form, so that the structure constants λ^{abc} are completely antisymmetric and the Casimir invariant $C_2 = \lambda^{abc} \lambda^{acb}$ equals $-N$. Moreover, we have

$$(T_n^a)^* = -T_{-n}^a.$$

The fermionic Hilbert space \mathcal{H}_f carries an irreducible representation of the canonical anticommutation relations (CAR)

$$\{\psi_n^a, \psi_m^b\} = 2\delta^{ab} \delta_{n,-m}$$

where $(\psi_n^a)^* = \psi_{-n}^a$. The Fock vacuum is a subspace of \mathcal{H}_f of dimension $2^{\lfloor \frac{N}{2} \rfloor}$. It carries an irreducible representation of the Clifford algebra spanned by the zero modes ψ_0^a and lies in the kernel of all ψ_n^a with $n < 0$. The loop algebra $\widehat{L\mathfrak{g}}$ acts in \mathcal{H}_f through the minimal representation of level h^\vee , the dual Coxeter number of G . The operators are explicitly realized as bilinears in the Clifford generators

$$K_n^a = -\frac{1}{4} \lambda^{abc} : \psi_{n-m}^b \psi_m^c :$$

and satisfy

$$[K_n^a, K_m^b] = \lambda^{abc} K_{n+m}^c + \frac{h^\vee}{4} n \delta^{ab} \delta_{n,-m}.$$

The normal ordering $::$ indicates that operators with positive Fourier index are placed to the left of those with negative index. In case of fermions there is a change of sign, $:: \psi_{-n}^a \psi_n^b := -\psi_n^b \psi_{-n}^a$ if $n > 0$. The full Hilbert space $\mathcal{H} = \mathcal{H}_b \otimes \mathcal{H}_f$ carries a tensor product representation of $\widehat{L\mathfrak{g}}$ of level $k + h^\vee$, with generators $S_n^a = T_n^a \otimes \mathbf{1} + \mathbf{1} \otimes K_n^a$.

The supercharge operator is defined by

$$Q = i\psi_n^a \left(T_{-n}^a + \frac{1}{3} K_{-n}^a \right)$$

and squares to the free Hamilton operator $h = Q^2$ of the supersymmetric WZW model

$$h = - : T_n^a T_{-n}^a : + \frac{\bar{k}}{2} : n \psi_n^a \psi_{-n}^a : + \frac{N}{24} = h_b + 2\bar{k}h_f + \frac{N}{24}$$

where $\bar{k} = \frac{k+h^\vee}{4}$. Interaction with external \mathfrak{g} -valued 1-forms A on the circle is introduced by minimal coupling

$$Q(A) = Q + i\bar{k} \psi_n^a A_{-n}^a$$

where A_n^a are the Fourier coefficients of A in the basis T_n^a , satisfying $(A_n^a)^* = -A_{-n}^a$. This provides us with a family of self-adjoint Fredholm operators $Q(A)$ that is equivariant with respect to the action of \widehat{LG} ,

$$S(g)^{-1}Q(A)S(g) = Q(A^g)$$

with $A^g = g^{-1}Ag + g^{-1}dg$. Infinitesimally this translates to

$$[S_n^a, Q(A)] = i\bar{k}(n\psi_n^a + \lambda^{abc}\psi_{n+m}^c A_{-m}^b) = -\mathcal{L}_n^a Q(A) .$$

where \mathcal{L}_n^a denotes the Lie derivative (infinitesimal gauge transformation) in the direction $X = S_n^a$. The interacting Hamiltonian $h(A) = Q(A)^2$ is given by

$$h(A) = h - \bar{k} (2S_n^a A_{-n}^a + \bar{k} A_n^a A_{-n}^a) = h + h_{int} .$$

Next we consider extending this construction to the fractional setting. As previously mentioned, this necessarily entails certain regularization. Let us first denote by S_0 the representation of the smooth loop algebra with commutation relations

$$[S_0(X), S_0(Y)] = S_0([X, Y]) + c_0(X, Y) .$$

We proceed by adding a 1-cochain

$$\begin{aligned} S(X) &= S_0(X) + \eta(X; B) \\ Q &= Q_0 + \eta(\psi; B) \end{aligned}$$

where in component notation $\eta(\psi; B) = \psi_n^a \eta(T_{-n}^a; B) = \psi_n^a \eta_{-n}^a$ and we denote the original supercharge by Q_0 from now on. Here $B = g^{-1}[\epsilon, g]$ parametrizes points on the Grassmannian, as in Section 4, with $g \in L_q G$. We write also $\mathcal{L}_X = X_n^a \mathcal{L}_{-n}^a$ for an element $X \in L_q \mathfrak{g}$. Adding cochains of the type η , which are functions of the variable B , means that we are extending the original loop algebra by Fréchet differentiable functions of B . Since gauge transformations are acting on B by the formula $B \mapsto g^{-1}Bg + g^{-1}[\epsilon, g]$, or infinitesimally as $B \mapsto [B, X] + [\epsilon, X]$ for $X \in L_q \mathfrak{g}$, the commutator of $S(X)$ by any Fréchet differentiable function f of B is given as

$$[S(X), f(B)] = \mathcal{L}_X f(B) .$$

The other new commutation relations will be

$$\begin{aligned} [S(X), S(Y)] &= S([X, Y]) + c(X, Y; B) \\ [S(X), \psi(Y)] &= \psi([X, Y]) \\ [S(X), Q] &= ic(X, \psi; B) \\ \{Q, \psi(Y)\} &= 2iS(Y) \end{aligned}$$

where $c(X, Y; B) = c_0(X, Y) + (\delta\eta)(X, Y; B)$ converges for an appropriate choice of η , according to Theorem 2. Moreover, we set $h = Q^2$ where

$$Q^2 = Q_0^2 - 2S(\eta) - \eta^2 + i\mathcal{L}_\psi \eta(\psi; B)$$

with Q_0 the old supercharge operator and $\eta^2 = \eta_n^a \eta_{-n}^a$ and $S(\eta) = \eta_n^a S_{-n}^a$. In Fourier basis, the whole algebra generated by $\{\psi_n^a, S_m^b, Q, h\}$ satisfies

$$\begin{aligned}
\{\psi_n^a, \psi_m^b\} &= 2\delta^{ab}\delta_{n,-m} \\
[S_n^a, S_m^b] &= \lambda^{abc}S_{n+m}^c + c_{n,m}^{a,b}(B) \\
[S_n^a, \psi_m^b] &= \lambda^{abc}\psi_{n+m}^c \\
\{\psi_n^a, Q\} &= 2iS_n^a \\
[S_n^a, Q] &= ic_{n,-m}^{a,b}(B)\psi_m^b \\
[\psi_n^a, h] &= 2c_{n,-m}^{a,b}(B)\psi_m^b \\
[h, S_n^a] &= 2S_m^b c_{n,-m}^{a,b}(B) - \psi_m^b \psi_p^c \mathcal{L}_{-p}^c c_{n,-m}^{a,b}(B) \\
[Q, h] &= 0
\end{aligned}$$

where $c_{n,m}^{a,b}(B) = c(S_n^a, S_m^b; B)$. For any Fréchet differentiable function $f = f(B)$,

$$\begin{aligned}
[S_n^a, f] &= \mathcal{L}_n^a f \\
[Q, f] &= i\psi_n^a \mathcal{L}_{-n}^a f \\
[h, f] &= -2S_n^a \mathcal{L}_{-n}^a f + \psi_n^a \psi_q^d \mathcal{L}_{-q}^d \mathcal{L}_{-n}^a f .
\end{aligned}$$

In the smooth case, $c_{n,m}^{a,b}(B) = kn\delta^{ab}\delta_{n,-m}$, one recovers the corresponding subalgebra of the superconformal current algebra [7]. Let us consider highest weight representations of the loop algebra generated by S and S_0 respectively. Since they differ by a coboundary, one can explicitly relate their vacua by restricting to the subalgebra of smooth loops $L\mathfrak{g} \subset L_q\mathfrak{g}$. Indeed, we have

$$S(X)|\Omega > = 0, \quad S_0(X)|\Omega_0 > = 0$$

for all $X \in L\mathfrak{g}_-$, which implies

$$S(X)|\Omega_0 > = (S_0(X) + \eta(X; B))|\Omega_0 > = \eta(X; B)|\Omega_0 > \neq 0 .$$

However if $(\delta\eta)(X, Y; B) = 0$ for all $X, Y \in L\mathfrak{g}_-$, then η restricts to a 1-cocycle on $L\mathfrak{g}_-$ and can be written $\eta(X; B) = \mathcal{L}_X \Phi(B)$ for some function Φ of the variable B on the smooth Grassmannian consisting of points $g^{-1}[\epsilon, g]$ for $g \in LG$. In fact, for the cochain η in Theorem 2 one can choose $\Phi(B) \sim \text{Tr}(\epsilon B^{2p+1})$ and the vacua are linked according to

$$|\Omega > = e^{-\Phi(B)}|\Omega_0 >$$

since for all $X \in L\mathfrak{g}_-$ we have

$$\begin{aligned}
S(X)|\Omega > &= e^{-\Phi(B)}(S(X) - \mathcal{L}_X \Phi(B))|\Omega_0 > \\
&= e^{-\Phi(B)}(S_0(X) + \eta(X; B) - \mathcal{L}_X \Phi(B))|\Omega_0 > = 0 .
\end{aligned}$$

6. TWISTED K-THEORY AND THE GROUP $L_q G$

We want to make sense of a family of supercharges $Q(A)$ which transform equivariantly under the action of the abelian extension $\widehat{L_q G}$ of the fractional loop group $L_q G$. This should generalize the construction of the similar family in the case of central extension of the smooth loop group. Let us recall the relevance of the latter for twisted K-theory over G of level $k + h^\vee$. We fix G to be a simple compact Lie group throughout this section. One can think of elements in $K^*(G, k + h^\vee)$ as maps $f : \mathcal{A} \rightarrow \text{Fred}(\mathcal{H})$, to Fredholm operators in a Hilbert space \mathcal{H} , with the property $f(A^g) = \hat{g}^{-1} f(A) \hat{g}$. Here $g \in LG$ and \mathcal{A} is the space of smooth \mathfrak{g} -valued vector potentials on the circle. The moduli space $\mathcal{A}/\Omega G$ (where ΩG is the group of based loops) can be identified as G . Actually, one can still use the equivariantness under constant loops so that we really deal with the case of G -equivariant twisted K-theory $K_G^*(G, k + h^\vee)$. For odd/even dimensional groups one gets elements in K^1/K^0 .

The real motivation here is to try to understand the corresponding supercharge operator Q arising from Yang-Mills theory in higher dimensions. If M is a compact spin manifold the gauge group $\text{Map}(M, G)$ can be embedded in U_p for any $2p > \dim M$; this was used in [MR] for constructing a geometric realization for the extension of $\text{Map}(M, G)$ arising from quantization of chiral Dirac operators in background gauge fields. This is an analogy for our embedding of $L_q G$ in U_p for $p > 1/2q$, the index $1/2q$ playing the role of the dimension of M .

The more modest aim here is to show that there is a true family of Fredholm operators which transforms covariantly under $L_q G$. The operators are parametrized by 1-forms on $L_q G$ and generalize the family of Fredholm operators $Q(A)$ from the smooth setting to the fractional case. Hopefully, this will help us understand the renormalizations needed for the corresponding problem in gauge theory on a manifold M .

Let us denote by $L^c G$ the Banach-Lie group of continuous loops in G . The natural topology of $L^c G$ is the metric topology defined as

$$d(f, g) = \sup_{x \in S^1} d_G(f(x), g(x))$$

where d_G is the distance function on G determined by the Riemann metric. Local charts on $L^c G$ are given by the inverse of the exponential function; at any point $f_0 \in L^c G$ we can map a sufficiently small open ball around f_0 to an open ball at zero in $L^c \mathfrak{g}$ by $f \mapsto \log(f_0^{-1} f)$.

In the smooth version LG the topology is locally given by the topology on $L\mathfrak{g}$; the topology of the vector space $L\mathfrak{g}$ is defined by the family of seminorms $\|f\|_n =$

$\sup_{x \in S^1} |f^{(n)}(x)|$ for a fixed norm $|\cdot|$ on \mathfrak{g} . More precisely, we can define a family of distance functions on LG by

$$d_n(f, g) = \sup_{x \in S^1} |f^{(n)}(x) - g^{(n)}(x)| \text{ for } n > 0$$

and for $n = 0$,

$$d_0(f, g) = \sup_{x \in S^1} d_G(f(x), g(x)) ,$$

where $f^{(n)}$ is the n :th derivative with respect to the loop parameter: We identify the first derivative as a function with values in \mathfrak{g} by left translation $f' \mapsto f^{-1}f'$ and then all the higher derivatives are \mathfrak{g} -valued functions on the circle.

The metric is then defined as

$$d(f, g) = \sum_{n \geq 0} \frac{d_n(f, g)}{1 + d_n(f, g)} 2^{-n} .$$

The subgroup $LG \subset L^cG$ is dense in the topology of the latter. For this reason the cohomology of L^cG is completely determined by restriction to LG . Actually, we have a stronger statement:

Lemma 6.1. (*Carey-Crowley-Murray*) *The group L^cG of continuous loops is homotopy equivalent to the smooth loop group LG .*

Proof. We may assume that G is connected; otherwise, one repeats the proof for each component of G . When G is connected the full loop group is a product of G and the group ΩG of based loops (whether continuous, smooth, or of type L_q). So we restrict to the group of based loops. As shown in [1], the groups Ω^cG and ΩG are weakly homotopic, i.e. the inclusion $\Omega G \subset \Omega^cG$ induces an isomorphism of the homotopy groups. According to the Theorem 15 by R. Palais [15] a weak homotopy equivalence of metrizable manifolds implies homotopy equivalence. Actually, in [1] the authors use the CW property of the loop groups for the last step. The CW property in the case of Ω^cG is a direct consequence of Theorem 3 in [14]. \square

Lemma 6.2. *The group L_qG is homotopy equivalent to LG and thus also to L^cG .*

Proof. The proof in [1] can be directly adapted from the smooth setting to the larger group L_qG . Let M be a compact manifold with base point m_0 and $C((I^n, \partial I^n), (M, m_0))$ the set of continuous maps from the n -dimensional unit cube to M such that the boundary of the cube is mapped to m_0 . The key step in their proof is the observation that in the homotopy class of any map $g \in C((I^n, \partial I^n), (M, m_0))$ there exists a smooth map; in addition, a homotopy can be given in terms of a differentiable map. Taking

$M = G$ and thinking of g as a representative for an element in the homotopy group $\pi_{n-1}(\Omega^c G)$. Since g is homotopic to a smooth map, it also represents an element in the smooth homotopy group of ΩG and thus also an element in the $(n-1)$:th homotopy group of Ω_q . In addition, a continuous homotopy is equivalent to a smooth homotopy. The embedding of $LG \subset L_q G$ is continuous in their respective topologies [this follows from the Sobolev norm estimates in the proof of Proposition 3.1] and therefore the representatives for the homotopy groups of the former are mapped to representatives of the homotopy groups of the latter. \square

Let $F : L^c G \rightarrow LG$ be a smooth homotopy equivalence. We define

$$\theta(f; g) = F(f)^{-1} F(fg)$$

for $f, g \in L^c G$. This is a 1-cocycle in the sense of

$$\theta(f; gg') = \theta(f; g)\theta(fg; g') .$$

For any $g \in LG$ we then have

$$\hat{\theta}(f; g)^{-1} Q_0 \hat{\theta}(f; g) = Q_0 + i\bar{k} \langle \psi, \theta(f; g)^{-1} \partial \theta(f; g) \rangle$$

where $\hat{\theta}$ is the lift of θ to the central extension \widehat{LG} .

Since the homotopy F is smooth we can define a Lie algebra cocycle

$$d\theta(f; X) = \frac{d}{dt} \Big|_{t=0} \theta(f; e^{tX})$$

with values in $L\mathfrak{g}$ for $X \in L^c \mathfrak{g}$. This is a 1-cocycle in the sense that

$$d\theta(f; [X, Y]) + \mathcal{L}_X d\theta(f; Y) - \mathcal{L}_Y d\theta(f; X) = [d\theta(f; X), d\theta(f; Y)] .$$

Let next $Q(A) = Q_0 + i\bar{k} \langle \psi, A \rangle$ be a perturbation of Q_0 by a function $A : L^c G \rightarrow L^c \mathfrak{g}$. The group $L^c G$ acts on A by right translation, $(g \cdot A)(f) = A(fg)$. Denote by $\hat{\Theta}(g)$ the operator consisting of the right translation on functions of f and of $\hat{\theta}(\cdot; g)$ acting on values of functions in the Hilbert space \mathcal{H} . Then

$$\hat{\Theta}(g)^{-1} Q(A) \hat{\Theta}(g) = Q(A^g) ,$$

where

$$(1) \quad (A^g)(f) = \theta(f; g)^{-1} A(fg) \theta(f; g) + \theta(f; g)^{-1} \partial \theta(f; g) .$$

Here ∂ is the differentiation with respect to the loop parameter. Since the group LG acts in \mathcal{H} through its central extension \widehat{LG} , the Lie algebra $L^c \mathfrak{g}$ acts through its abelian extension by $Map(L^c G, i\mathbb{R})$, the extension being defined by the 2-cocycle

$$\omega(f; X, Y) = [\widehat{d\theta}(f; X), \widehat{d\theta}(f; Y)] - \widehat{d\theta}(f; [X, Y]) - \mathcal{L}_X \widehat{d\theta}(f; Y) + \mathcal{L}_Y \widehat{d\theta}(f; X) .$$

For an infinitesimal gauge transformations X , the formula (4) leads to

$$(2) \quad \delta_X A = [A, \theta(f; X)] + \partial\theta(f; X) + \mathcal{L}_X A$$

which should be compared with $\delta_X A = [A, X] + \partial X$ in the smooth case, for constant functions $A : LG \rightarrow L\mathfrak{g}$.

The Lie algebra cohomology of any Lie algebra, with coefficients in the module of smooth functions on the simple compact Lie group G , is by definition the same as the de Rham cohomology of G . We apply this to the loop group LG together with the fact that the standard central extension of $L\mathfrak{g}$, when viewed as a left invariant 2-form on LG , generates $H^2(LG, \mathbb{R})$. Thus the Lie algebra cohomology of $L\mathfrak{g}$ with coefficients in the module of smooth functions on LG is one dimensional.

By the Lemma 6.2 we have

Lemma 6.3. *For a simple compact Lie group G the Lie algebra cohomology in degree 2 of $L^c\mathfrak{g}$ with coefficients in the module of smooth functions on L^cG (with respect to the Banach manifold structure) is one dimensional and the class of a cocycle is fixed by the restriction to the smooth version LG .*

Remark We can easily produce an explicit homotopy connecting the standard 2-cocycle on $L\mathfrak{g}$ to the restriction of the 2-cocycle on $L^c\mathfrak{g}$ to the smooth subalgebra $L\mathfrak{g}$. Let $F_s : L^cG \rightarrow LG$ be any one-parameter family of maps connecting the identity to the retraction on LG , $0 \leq s \leq 1$. For each s , $F_s(f)$ is smooth if f is smooth. Define

$$\eta = \frac{1}{2} \int_0^1 \text{tr} [\epsilon F_s^{-1} \delta F_s] F_s^{-1} \partial_s F_s .$$

Denote $c_0(X, Y) = \frac{1}{2} \text{tr} X[\epsilon, Y]$, the standard Lie algebra 2-cocycle on $L\mathfrak{g}$, and $c_1(X, Y) = \frac{1}{2} \text{tr} d\theta(f; X)[\epsilon, d\theta(f; Y)]$ the 2-cocycle coming from the homotopy $F = F_0$. Then one checks that, for the restriction to LG ,

$$c_0 - c_1 = \delta\eta .$$

Proposition 6.4. *The unitary subgroup $U_p \subset GL_p$ for each $p \geq 1$ is smoothly homotopic to U_1 .*

Proof. We prove the claim inductively by constructing a homotopy equivalence from U_p to $U_{p/2}$ for all $p \geq 2$. Let $g \in U_p$ with

$$g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} .$$

Let us denote $x = \alpha\gamma^* - \beta\delta^* \in L_{2p}$. Define the unitary operator $h(g)$

$$h(g) = \exp \begin{pmatrix} 0 & -x/2 \\ x^*/2 & 0 \end{pmatrix}.$$

Let $F(g) = h(g)^{-1}g$. Using $x^2 \in L_p$, by a direct computation one checks that the upper right block in this operator is equal to

$$\beta + \frac{1}{2}(\alpha\gamma^* - \beta\delta^*)\delta \bmod L_p.$$

Using the unitarity relations $\alpha^*\beta + \gamma^*\delta = 0$ and $\beta^*\beta + \delta^*\delta = 1$ we see that the above operator is equal to $\beta\beta^*\beta$. Since $\beta \in L_{2p}$, by the operator Hölder inequalities $\beta\beta^*\beta$ is in $L_{2p/3} \subset L_p$. Thus $F(g) \in U_{p/2}$. This map is homotopic to the identity map in U_p : one just needs to replace the operator x by tx where $0 \leq t \leq 1$. At $t = 0$ we then have $F_t(g) = g$. Since the blocks of $F(g)$ are rational functions in the blocks of g without singularities, the map $g \mapsto F(g)$ is smooth in the natural Banach manifold structures of the groups U_p and $U_{p/2}$. The same argument shows that the identity map on U_r contracts to a map to the subgroup $U_{p/2}$ for any $p \geq r \geq p/2$. \square

According to [PS] the restriction of the standard 2-cocycle of the Lie algebra of U_1 (denoted by U_{res} in [PS]) to the subalgebra $L\mathfrak{g}$ is the cocycle defining an affine Kac-Moody algebra. In combination with Lemma 6.3 and Proposition 6.4 we conclude:

Proposition 6.5. *The restriction from L^cG to L_qG of the Lie algebra cocycle ω is equivalent to the Lie algebra cocycle of Theorem 4.1 (obtained by restriction of c_p from GL_p to L_qG with $p \geq 1/2q$).*

Remark The previous Remark applies here, too. By a similar formula one can produce an explicit homotopy between the standard 2-cocycle on the Lie algebra of $U_1 = U_{res}$ and the cocycle obtained from the restriction from U_p to the subgroup U_1 , $p \geq 1$.

We close this section by showing that the transformation of the field A under L_qG has a more geometric interpretation.

Using the inner product $\langle X, Y \rangle = \int_{S^1} (X(x), Y(x)) dx$ in $L_q\mathfrak{g}$, where (\cdot, \cdot) is an invariant inner product in \mathfrak{g} , and the fact a Lie group is a parallelizable manifold we can think of A as a 1-form on the loop group L_qG . We denote by \mathcal{A} the space of all Fréchet differentiable 1-forms on L_qG . There is a circle bundle P over L_qG with connection and curvature; the curvature ω is given by the cocycle of the abelian extension, $\omega_g(X, Y) = c_p(X, Y; B)$ (for a fixed $p > 1/2q$) where the elements $X, Y \in T_g(L_qG)$ on the left are identified as left invariant vector fields (elements of the Lie algebra of L_qG) on

the right-hand-side of the equation. Here we have set $B = g^{-1}dg = g^{-1}[\epsilon, g]$. Let us denote by $\widehat{L_q G}$ the extension of $L_q G$ by the abelian normal subgroup $Map(L_q G, S^1)$ corresponding to the given Lie algebra extension.

Conversely, starting from the abelian extension $\widehat{L_q G}$, viewed as a principal bundle over $L_q G$ with fiber $Map(L_q G, S^1)$, we can recover the geometry of the circle bundle P . The connection in P is obtained as follows. First, the connection form in $\widehat{L_q G}$ is given as

$$\Psi = Ad_{\hat{g}}^{-1} \text{pr}_c(d\hat{g}\hat{g}^{-1})$$

where pr_c is the projection onto the abelian ideal $Map(L_q G, i\mathbb{R})$.

Remark In the case of the central extension \widehat{LG} of LG the adjoint action on $\text{pr}_c(\bullet)$ is trivial but in the case of the abelian extension of $L_q G$ it is needed in order to guarantee that the connection form is tautological in the vertical directions in the tangent bundle TP .

The connection ∇ on P is then defined using the identification of P as the subbundle of $\widehat{L_q G} \rightarrow L_q G$ with fiber S^1 consisting of constant functions in $Map(L_q G, S^1)$, that is, at each base point $g \in L_q G$ we have the homomorphism γ sending a function $f \in Map(L_q G, S^1)$ to the value of f at the neutral element. This induces a homomorphism $d\gamma : Map(L_q G, i\mathbb{R}) \rightarrow i\mathbb{R}$ defining the $i\mathbb{R}$ -valued 1-form $\psi = d\gamma \circ \Psi$, defining the covariant differentiation ∇ in the associated complex line bundle L .

An arbitrary connection in the bundle P is then written as a sum $\nabla + A$ with $A \in \mathcal{A}$.

Theorem 6.6. *The transformation of $\nabla + A$ under an infinitesimal right action of $(X, \alpha) \in \widehat{L_q \mathfrak{g}}$ induces the action $\delta A = \omega_g(X, \bullet) + \mathcal{L}_X A + \mathcal{L}_\bullet \alpha$. That is, under the infinitesimal transformation δ_X the transformation is the same as in (2). The function α can be interpreted as an infinitesimal gauge transformation in the line bundle L over $L_q G$.*

Proof. Let $(X, \alpha) \in \widehat{L_q \mathfrak{g}}$. Infinitesimally, the right action is generated by left invariant vector fields. Thus the infinitesimal shift in the direction (X, α) , of the connection evaluated at the tangent vector Y is the commutator $[\nabla_Y, \nabla_X + \alpha]$. The first term in the commutator is equal to the curvature of the circle bundle evaluated in the directions X, Y . This is in turn equal to $\omega_g(X, Y)$. Since the covariant derivative ∇_Y acts as the Lie derivative \mathcal{L}_Y on functions, we obtain

$$(3) \quad \delta_{(X, \alpha)}(\nabla + A) = \omega_g(X, \bullet) + \mathcal{L}_X A + \mathcal{L}_\bullet \alpha .$$

□

It follows that we can morally interpret $Q(A)$ as a Dirac operator on the loop group $L_q G$ twisted by a complex line bundle.

In the case of a central extension the above formulas (on level $k + \kappa$) reproduce the classical gauge action, after identification of the dual $L\mathbf{g}^*$ as $L\mathbf{g}$ using the Killing form,

$$(4) \quad A \mapsto [A, X] + \frac{1}{4}(k + h^\vee)dX,$$

for a left invariant 1-form A . Namely, the right Lie derivative in (3), when acting on left invariant forms, produces the commutator term $[A, X]$ and whereas the shift by dX is coming from the central extension $\omega_g(X, \bullet) = c_0(X, \bullet)$. The third term on the right-hand-side of the equation (3) is absent since in case of central extension we can take $\alpha = \text{constant}$.

Going the other way, starting from the action on \mathcal{A} in terms of the cocycle ω , we may take the restriction to the Lie algebra of smooth loops and the connection ∇ becomes a sum

$$\nabla = \nabla_0 + \eta$$

where ∇_0 is the connection in P defined by the standard central extension of the loop group and η is a 1-form on LG . This eta form is actually the 1-cochain relating the central extension to the abelian extension of LG . Writing now $a = \eta + A$ and $S(X) = S_0(X) + \eta(X)$ we recover the gauge action formula $a \mapsto [a, X] + \frac{1}{4}(k + h^\vee)dX$ with respect to $S_0(X)$. Thus the addition of η can be viewed as a renormalization needed for extending from the case of smooth loop algebra to the fractional loops.

ACKNOWLEDGEMENTS

The authors would like to thank the Erwin Schrödinger Institute for Mathematical Physics for hospitality where this work was initiated. The work of the second author was partially supported by the grant 516 8/07-08 from Academy of Finland.

APPENDIX: Proof of Theorem 1

Define a 1-cochain by

$$\tilde{\eta}_p(X; B) = \text{Str}\left(B^{2p+1}X\right)$$

for $p \geq 0$. Using Palais' formula the coboundary is given by

$$\begin{aligned} (\delta\tilde{\eta}_p)(X, Y; B) &= \sum_{k=0}^{2p} \text{Str} \left(B^k [B, X] B^{2p-k} Y - B^k [B, Y] B^{2p-k} X \right) + \\ &\quad + \sum_{k=0}^{2p} \text{Str} \left(B^k dX B^{2p-k} Y - B^k dY B^{2p-k} X \right) - \text{Str} \left(B^{2p+1} [X, Y] \right). \end{aligned}$$

By cyclicity of trace we can rewrite the first sum,

$$\begin{aligned} &\sum_{k=0}^{2p} \text{Str} \left(B^k [B, X] B^{2p-k} Y - B^k [B, Y] B^{2p-k} X \right) = \\ &= \text{Str} \left(B[X, B^{2p} Y] - B[Y, B^{2p} X] \right) + \sum_{k=1}^{2p} \text{Str} \left(B^k [B, X] B^{2p-k} Y - B^k [B, Y] B^{2p-k} X \right). \end{aligned}$$

This can be simplified further. Using $[a, bc] = [a, b]c + b[a, c]$ repeatedly we get

$$B[X, B^{2p} Y] = B^{2p+1} [X, Y] + \sum_{k=0}^{2p-1} B^{2p-k} [X, B] B^k Y$$

and inserting this into the expression above yields

$$\begin{aligned} &2\text{Str} \left(B^{2p+1} [X, Y] \right) + \sum_{k=0}^{2p-1} \text{Str} \left(B^{2p-k} [X, B] B^k Y - B^{2p-k} [Y, B] B^k X \right) + \\ &+ \sum_{k=1}^{2p} \text{Str} \left(B^k [B, X] B^{2p-k} Y - B^k [B, Y] B^{2p-k} X \right) = 2\text{Str} \left(B^{2p+1} [X, Y] \right). \end{aligned}$$

Thus the expression for the coboundary $\delta\tilde{\eta}_p$ reduces to

$$(\delta\tilde{\eta}_p)(X, Y; B) = \text{Str} \left(B^{2p+1} [X, Y] + \sum_{k=0}^{2p} B^k dX B^{2p-k} Y - B^k dY B^{2p-k} X \right).$$

We split the remaining sum into even and odd powers. When $k = 2m$ is even,

$$\begin{aligned} &\sum_{m=0}^p \text{Str} \left(B^{2m} dX B^{2p-2m} Y - B^{2m} dY B^{2p-2m} X \right) = \\ &= \sum_{m=0}^p \text{Str} \left(B^{2m} dX B^{2p-2m} Y + B^{2m} Y B^{2p-2m} dX \right) \end{aligned}$$

and when $k = 2m - 1$ is odd,

$$\begin{aligned} &\sum_{m=1}^p \text{Str} \left(B^{2m-1} dX B^{2p-2m+1} Y - B^{2m-1} dY B^{2p-2m+1} X \right) = \\ &= \sum_{m=1}^p \text{Str} \left(B^{2m-1} X B^{2p-2m+2} Y - B^{2m} X B^{2p-2m+1} Y \right) \end{aligned}$$

where we have used $dB^{2m} = 0$, $dB^{2m+1} = -B^{2m+2}$ and “integration by parts”

$$\begin{aligned} d(B^{2m-1}XB^{2p-2m+1}Y) &= -B^{2m}XB^{2p-2m+1}Y - B^{2m-1}dXB^{2p-2m+1}Y + \\ &\quad + B^{2m-1}XB^{2p-2m+2}Y + B^{2m-1}XB^{2p-2m+1}dY. \end{aligned}$$

Finally we show that the sum in $\tilde{c}_p(X, Y; B)$ for values $k \geq 1$ equals, up to the normalization factor 2^{2p} , the odd part of the sum above;

$$\begin{aligned} &\sum_{k=1}^p (-1)^k \text{Str} \left(B^{2p-k+1}XB^kY - B^{2p-k+1}YB^kX \right) = \\ &= \sum_{m=1}^{p/2} \text{Str} \left(B^{2p-2m+2}YB^{2m-1}X - B^{2p-2m+2}XB^{2m-1}Y \right) + \\ &\quad + \sum_{m=1}^{p/2} \text{Str} \left(B^{2p-2m+1}XB^{2m}Y - B^{2p-2m+1}YB^{2m}X \right) = \\ &= \sum_{m=1}^{p/2} \text{Str} \left(B^{2m-1}XB^{2p-2m+2}Y - B^{2m}XB^{2p-2m+1}Y \right) + \\ &\quad + \sum_{m=1}^{p/2} \text{Str} \left(B^{2p-2m+1}XB^{2m}Y - B^{2p-2m+2}XB^{2m-1}Y \right). \end{aligned}$$

Shifting the index $m = n - \frac{p}{2}$ in the second sum we get

$$\sum_{n=(p+2)/2}^p \text{Str} \left(B^{3p-2n+1}XB^{2n-p}Y - B^{3p-2n+2}XB^{2n-p-1}Y \right)$$

and reversing the summation order using

$$\sum_{\alpha}^{\beta} x^{\pm 2n} = \sum_{\alpha}^{\beta} x^{\pm 2(\alpha+\beta) \mp 2n}$$

this becomes

$$\sum_{n=(p+2)/2}^p \text{Str} \left(B^{2n-1}XB^{2p-2n+2}Y - B^{2n}XB^{2p-2n+1}Y \right).$$

Altogether this is precisely the odd part of the sum above. Now we define a new cocycle

$$\begin{aligned} c_p(X, Y; B) &= \tilde{c}_p(X, Y; B) - 2^{2p}(\delta\tilde{\eta}_p)(X, Y; B) \\ &= -2^{2p} \sum_{m=0}^p \text{Str} \left(B^{2m}dXB^{2p-2m}Y - B^{2m}dYB^{2p-2m}X \right) \\ &= -2^{2p} \sum_{m=0}^p \text{Str} \left(B^{2m}dXB^{2p-2m}Y + B^{2m}YB^{2p-2m}dX \right). \end{aligned}$$

We note that this cocycle vanishes when both X, Y have only negative (or positive) Fourier components, because the trace of $B^n dX B^{2m} dY$ and $B^{2m} dX B^{2n} dY$ are zero when X, Y are upper (or lower) triangular matrices with respect to the ϵ grading.

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